Sampled-data-based LQ control of stochastic linear continuous-time systems

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Abstract Sampled-data (SD) based linear quadratic (LQ) control problem of stochastic linear continuous-time (LCT) systems is discussed. Two types of systems are involved. One is time-invariant and the other is time-varying. In addition to stability analysis of the closed-loop systems, the index difference between SD-based LQ control and conventional LQ control is investigated. It is shown that when sample time ΔT is small, so is the index difference. In addition, the upper bounds of the differences are also presented, which are $O(\Delta T^2)$ and $O(\Delta T)$, respectively.

Keywords: **stochastic system, sampled-data-based control, time-varying parameter, LQ index.**

Information on system structure, states, outputs etc. is very important and indispensable when we implement a control to the system involved. However, sometimes not all the information is available because of the comprehension degree to the system and the impact and restriction of many factors such as the measure instruments, the estimation methods, the sampling and computation speed. For instance, in some cases, the system state itself is a continuous process, but what we can measure is only sampled data, due to the limit of the sensor, and the speed of sample pattern and signal processing. This leads to the following questions: for a given control criterion, under what condition can a sampled-data (SD) based control match a conventional full-state (FS) based control? If the former does not match the latter, what is the difference? Can the difference be expressed quantitatively?

Stimulated by the above-mentioned issues, this paper studies the quadratic optimal control problem of stochastic linear continuous-time systems (LCT). Two types of such systems are investigated. One is time-invariant and the other is with time-varying Markovian jump parameters. In this work, the state is governed by a stochastic LCT differential equation, and thus, is a continuous process; but the state is unmeasurable except at the discrete sample time instants. So, only the discrete information (i.e. sampled data) is available to the designer for control design. Such control is usually named SD-based control. Generally speaking, the closed-loop system with SD-based control is a hybrid system with both continuous and discrete information. There has been lots of literature on this kind of system (see e.g. refs. [1—6]). The objects studied in refs. [1—4] were deterministic systems; and the control design method was first to discretize the original continuous-time model, and then, design the optimal control for the discrete model. Just as shown by Example 6.6.1 of ref. [4], the disadvantage of this method is that the control effect at the sample times is overemphasized, which is liable to cause the system fluctuating between the sample times. Refs. [5, 6] considered SD-based stabilization control of nonlinear stochastic systems with input gain as a known identity matrix. A relationship between sample step size and stability property of the closed-loop system was established. The contribution of this paper is: (i) General stochastic LCT systems are discussed, including systems with time-invariant and time-varying Markovian jump parameters. (ii) The control design is directly based on the original continuous system and the original continuous performance index, without involving any discretized models and discretized indices. (iii) In addition to stability of the closed-loop systems, the optimal index is analyzed. Particularly, the performance indices corresponding to the SD-based LQ control and the FS-based LQ control, respectively, are compared. And the difference between the performance indices is quantitatively expressed.

1 SD-based LQ optimal control of time-invariant systems

In this section, we consider SD-based LQ optimal control problem of LCT system with time-invariant parameters. Suppose the system model is of the form

$$dx_t = Ax_t dt + Bu_t dt + CdW_t, \tag{1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ and $W_t \in \mathbb{R}^n$ are system state, input and disturbance, respectively; and W_t is a standard Brownian motion.

Suppose the parameters A, B are known, and $[A, B]$ is controllable. The quadratic index function is

$$J(u) = \limsup_{t\to\infty} \frac{1}{t}\int_0^t (x_s^\tau Q x_s + u_s^\tau R u_s)ds, \tag{2}$$

where $R > 0$, $Q \geqslant 0$, and $[A, Q^{1/2}]$ is observable.

Our objectives are (a) to investigate the effect of the sample step size on the control functions, (b) to find out the stabilizability condition by the SD-based control, (c) to study the difference between the optimal index values corresponding to the SD-based LQ control and the FS-based LQ control, respectively, and (d) to establish an explicit expression of this index difference. To get a comparative picture, we first recall the optimal index value corresponding to the FS-based LQ control in Theorem 1.1, and then, present the optimal index value corresponding to the SD-based LQ control in Theorem 1.2.

For the convenience of citation, we denote $u = \{u_t,\, t \geqslant 0\}$, $u^* = \{u_t^*,\, t \geqslant 0\}$, and introduce the following admissible control set:

$$\begin{aligned} \mathcal{U} = \{u :\ & u_t \in \sigma\{x_s : s \leqslant t\} \text{ such that the state } \{x_t\} \text{ of (1) satisfying} \\ & \lim_{t\to\infty} \frac{1}{t}\|x_t\|^2 = 0 \text{ and } \limsup_{t\to\infty} \frac{1}{t}\int_0^t \|x_s\|^2 ds < \infty\}. \end{aligned} \tag{3}$$

Theorem 1.1. Consider system (1). Assume $[A, B, Q^{1/2}]$ is controllable and observable, P is the unique positive-definite solution of the following algebraic Riccati equation

$$A^\tau P + PA^\tau - PBR^{-1}B^\tau P + Q = 0. \tag{4}$$

Then

(i) For any $u \in \mathcal{U}$, $J(u) \geqslant \mathrm{tr}(C^\tau PC)$.

(ii) $u_t^* \triangleq -R^{-1}B^\tau Px_t \in \mathcal{U}$, and $J(u^*) = \mathrm{tr}(C^\tau PC) = \min\limits_{u\in\mathcal{U}} J(u)$.

This result can be found in some works (e.g. ref. [7]). And so, the proof is omitted here.

We now consider the SD-based LQ optimal control. Assume the sample step size of the system signal is ΔT. By using a zero-order hold, we can design an SD-based LQ control as

$$u_t = -R^{-1}B^\tau P x_{k\Delta T}, \quad t \in [k\Delta T, (k+1)\Delta T). \tag{5}$$

For simplicity, we will use x_k to denote the sampled data $x_{k\Delta T}$ of the state x at the sample time $k\Delta T$.

Under the control law (5), we have the following results.

Theorem 1.2. Consider the system (1). Assume $[A, B, Q^{1/2}]$ is controllable and observable. Under the SD-based LQ control (5), if ΔT satisfies

$$\Delta T e^{\|A\|\Delta T} \leqslant \frac{1}{(1+3\|B_1^\tau H\|)\|A_1\|}, \tag{6}$$

then

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t (x_s^\tau Q x_s + u_s^\tau R u_s) ds \leqslant \mathrm{tr}(C^\tau P C) + O(\Delta T^2), \tag{7}$$

where

$$A_1 \triangleq A - BR^{-1}B^\tau P, \quad B_1 \triangleq BR^{-1}B^\tau P, \quad H = \int_0^\infty e^{A_1^\tau t} e^{A_1 t} dt. \tag{8}$$

Remark 1.1. Let $c_1 = \Delta T \|A_1\| e^{\|A\|\Delta T}$, $\alpha_1 = \dfrac{\|B_1^\tau H\| c_1}{1-c_1}$. Then when ΔT satisfies (6), it is easy to verify that

$$c_1 \in (0,1), \quad \alpha_1 \in \left(0, \ \frac{1}{3}\right].$$

Remark 1.2. Let $f(x) = xe^{\|A\|x}$. It is easy to see that, for $x \in [0, \infty)$, $f'(x) = (1 + \|A\|x)e^{\|A\|x} \geqslant 1 > 0$. This implies that $f(x) = xe^{\|A\|x}$ is strictly increased on $x \in [0, \infty)$. Therefore, the range of ΔT can simply be determined as:

$$0 < \Delta T \leqslant \min\left\{\alpha, \ \frac{1}{(1+3\|B_1^\tau H\|)\|A_1\| e^{\|A\|\alpha}}\right\},$$

where α is any given positive real number. In particular, one can take $\alpha = 1$. In this case,

$$0 < \Delta T \leqslant \min\left\{1, \ \frac{1}{(1+3\|B_1^\tau H\|)\|A_1\| e^{\|A\|}}\right\}.$$

Of course, if the solution x_0 of equation $xe^{\|A\|x} = \frac{1}{(1+3\|B_1^\tau H\|)\|A_1\|}$ is given, then the range of ΔT can be determined by: $\Delta T \in (0, x_0)$. Unfortunately, generally speaking, solving this equation needs lots of computational load.

Remark 1.3. Here we give only a range of ΔT. When the sample step size ΔT is in this range, it can be shown that the SD-based LQ control (5) guarantees the stability of the closed-loop system, and is suboptimal. For general nonlinear systems or other control objectives, it is difficult and complex to solve the following problems, such as, how to figure out the maximal range of ΔT, how to choose ΔT optimally, and how to obtain an explicit expression describing the relationship between the sample step size and system structure and parameters. The best way is to study case by case. This is because these problems depend not only on system structure and parameters, but also on the control objectives. For instance, the control objectives of ref. [6] are system stabilization and control robustness. By an appropriate description of the uncertainty measure, a relationship of the sample step size to the uncertainty and stabilizability of the system is established. However the control objectives of this paper are system stabilization and SD-based suboptimal LQ control. Hence, the choice of the sample step size depends on not only the system parameters A,B, but also the index parameters R, Q and the index's quadratic form. For instance, P,A_1,B_1 and H stem from the index. Furthermore, since $H = \int_0^\infty e^{A_1^\tau t} e^{A_1 t} dt$, the value of $\|H\|$ is, in general, dependent on the real part of A_1: the smaller the real part of A_1 is, the larger $\|H\|$ is. By (6), this may reduce the range of ΔT.

We first introduce the following lemma before proving Theorem 1.2. This lemma is critical in the proofs of Theorem 1.2, Theorem 2.1 and Theorem 2.2 of the next section.

Lemma 1.1[8]**.** Let $\{x_t, \mathcal{F}_t\}$ be an adaptive process such that

$$\int_0^t x_s^2 ds < \infty \quad \text{a.s.} \quad \forall t \geqslant 0.$$

If $\{w_t, \mathcal{F}_t\}$ is a Wiener process, then as $t \to \infty$

$$\int_0^t x_s dw_s = O\left(\sqrt{\left(\int_0^t x_s^2 ds\right) \log\log\left(e + \int_0^t x_s^2 ds\right)}\right) \quad \text{a.s.}$$

Proof of Theorem 1.2. For the simplicity of expression, let us introduce the following symbol $t' \triangleq \left\lceil \frac{t}{\Delta T} \right\rceil \Delta T$. Here, $\lceil x \rceil$ denotes the maximal integer less than or equal to x.

Under the SD-based control (5), system (1) has the following closed-loop form

$$\begin{aligned} dx_t &= A_1 x_t dt + B_1(x_t - x_{t'})dt + CdW_t & (9)\\ &= A_1 x_{t'} dt + A(x_t - x_{t'})dt + CdW_t. & (10) \end{aligned}$$

By Ito's formula, $\forall t \in [t', t' + \Delta T)$,

$$x_t - x_{t'} = A \int_{t'}^t (x_s - x_{s'})ds + (t - t')A_1 x_{t'} + C(W_t - W_{t'}). \tag{11}$$

By (11), $\forall t \in [t', t' + \Delta T)$,

$$\|x_t - x_{t'}\| \leqslant \|A\| \int_{t'}^t \|x_s - x_{s'}\| ds + \Delta T \|A_1\| \|x_{t'}\| + \|C(W_t - W_{t'})\|. \tag{12}$$

Applying Grownwall lemma, we obtain

$$\begin{aligned} \|x_t - x_{t'}\| &\leqslant \Delta T \|A_1\| \|x_{t'}\| e^{\|A\|(t-t')} + \|C(W_t - W_{t'})\| \\ &\quad + \|A\| \int_{t'}^t e^{\|A\|(t-s)} \|C(W_s - W_{t'})\| ds \leqslant c_1 \|x_{t'}\| + c_2(t), \end{aligned} \tag{13}$$

where

$$\begin{cases} c_1 = \Delta T \|A_1\| e^{\|A\| \Delta T}, \\ c_2(t) = \|C(W_t - W_{t'})\| + \|A\| e^{\|A\| \Delta T} \int_{t'}^t \|C(W_s - W_{t'})\| ds. \end{cases}$$

Notice that by (9) and Ito's formula,

$$\begin{aligned} x_t^\tau P x_t - x_0^\tau P x_0 = &\int_0^t x_s^\tau (A_1^\tau P + P A_1) x_s ds + \int_0^t ((x_s - x_{s'})^\tau B_1^\tau P x_s + x_s^\tau P B_1 (x_s - x_{s'}))ds \\ &+ 2 \int_0^t x_s^\tau P C dW_s + t \cdot \mathrm{tr}(C^\tau P C). \end{aligned}$$

Then, we have

$$
\begin{aligned}
&x_t^\tau P x_t+\int_0^t(x_s^\tau Q x_s+u_s^\tau R u_s)ds\\
&=x_0^\tau P x_0+\int_0^t x_s^\tau(A_1^\tau P+PA_1)x_s ds+\int_0^t(x_s^\tau Q x_s+(u_s^*+u_s-u_s^*)^\tau R(u_s^*+u_s-u_s^*))ds\\
&\quad+\int_0^t((x_s-x_{s'})^\tau B_1^\tau P x_s+x_s^\tau P B_1(x_s-x_{s'}))ds+2\int_0^t x_s^\tau P C dW_s+t\cdot\mathrm{tr}(C^\tau P C)\\
&=x_0^\tau P x_0+\int_0^t(x_s-x_{s'})^\tau PBR^{-1}B^\tau P(x_s-x_{s'})ds+2\int_0^t x_s^\tau P C dW_s+t\cdot\mathrm{tr}(C^\tau P C)\\
&=x_0^\tau P x_0+\int_0^t\|x_s-x_{s'}\|_G^2 ds+2\int_0^t x_s^\tau P C dW_s+t\cdot\mathrm{tr}(C^\tau P C),
\end{aligned}\tag{14}
$$

where $G\stackrel{\triangle}{=}PBR^{-1}B^\tau P$, $\|x_s-x_{s'}\|_G^2=(x_s-x_{s'})^\tau G(x_s-x_{s'})$.

Now, we show that when ΔT is small,

$$
\limsup_t \frac{1}{t}\int_0^t\|x_{s'}\|^2 ds<\infty. \tag{15}
$$

It is easy to see that H satisfies $A_1^\tau H+HA_1=-I$. By (9) and Ito's formula,

$$
\begin{aligned}
x_t^\tau H x_t=x_0^\tau H x_0-\int_0^t\|x_s\|^2 ds+2\int_0^t(x_s-x_{s'})^\tau B_1^\tau H x_s ds+t\cdot\mathrm{tr}(C^\tau H C)\\
+2\int_0^t x_s^\tau H C dW_s.
\end{aligned}\tag{16}
$$

Further, by (13),

$$
\begin{aligned}
(x_s-x_{s'})^\tau B_1^\tau H x_s&\leqslant\|B_1^\tau H\|\cdot\|x_s-x_{s'}\|\cdot\|x_s\|\leqslant\|B_1^\tau H\|(c_1\|x_{s'}\|+c_2(s))\|x_s\|\\
&\leqslant\|B_1^\tau H\|\left[\frac{c_1}{1-c_1}\|x_s\|+\frac{c_2(s)}{1-c_1}\right]\|x_s\|\leqslant\alpha_1\|x_s\|^2+\beta_1(s)\|x_s\|,
\end{aligned}\tag{17}
$$

where

$$
\alpha_1=\frac{\|B_1^\tau H\|c_1}{1-c_1},\quad \beta_1(t)=\frac{\|B_1^\tau H\|c_2(t)}{1-c_1}.
$$

Here we have used $c_1\in(0,1)$ (see Remark 1.1) and the following inequality induced from (13)

$$
\|x_{s'}\|\leqslant\frac{1}{1-c_1}\|x_s\|+\frac{c_2(s)}{1-c_1}. \tag{18}
$$

Substituting (17) into (16), and using Lemma 1.1, we have

$$
\begin{aligned}
x_t^\tau H x_t&\leqslant x_0^\tau H x_0-(1-2\alpha_1)\int_0^t\|x_s\|^2 ds+2\int_0^t\beta_1(s)\|x_s\|ds\\
&\quad+t\cdot\mathrm{tr}(C^\tau H C)+2\int_0^t x_s^\tau H C dW_s\\
&\leqslant x_0^\tau H x_0-(1-2\alpha_1)\int_0^t\|x_s\|^2 ds+2\left(\int_0^t\|\beta_1(s)\|^2 ds\right)^{1/2}\left(\int_0^t\|x_s\|^2 ds\right)^{1/2}\\
&\quad+t\cdot\mathrm{tr}(C^\tau H C)+O\left(\left(\int_0^t\|x_s\|^2 ds\right)^{2/3}\right).
\end{aligned}\tag{19}
$$

From (6) and Remark 1.1 it follows that

$$
1-2\alpha_1\geqslant\frac{1}{3}.
$$

Then by the independent increment property of $\|W_t - W_{t'}\|$ and the large number law, we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \|\beta_1(s)\|^2 ds = E\int_0^{\Delta T}\|\beta_1(s)\|^2 ds = O(\Delta T^2). \tag{20}$$

And so,

$$\begin{aligned}&\limsup_{t\to\infty}\left(\frac{1}{t}x_t^\tau H x_t + (1-2\alpha_1)\frac{1}{t}\int_0^t \|x_s\|^2 ds\right)\\ \leqslant\ &\mathrm{tr}(C^\tau HC) + O\left(\limsup_{t\to\infty}\left(\frac{1}{t}\int_0^t\|x_s\|^2 ds\right)^{1/2}\right) + O\left(\limsup_{t\to\infty}\frac{1}{t}\left(\int_0^t \|x_s\|^2 ds\right)^{2/3}\right).\end{aligned}$$

Thus,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \|x_s\|^2 ds < \infty. \tag{21}$$

Furthermore, by (18) and (21) it can be seen that (15) is true. Again, by (13) we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\int_0^t \|x_s - x_{s'}\|_G^2 ds &\leqslant 2c_1^2\cdot\limsup_{t\to\infty}\frac{1}{t}\int_0^t\|x_{s'}\|_G^2 ds + 2\limsup_{t\to\infty}\frac{1}{t}\int_0^t c_2(s)^2 ds\\ &= O(\Delta T^2).\end{aligned} \tag{22}$$

This together with (14) gives

$$\begin{aligned}&\limsup_{t\to\infty}\left[\frac{1}{t}x_t^\tau P x_t + \frac{1}{t}\int_0^t (x_s^\tau Q x_s + u_s^\tau R u_s) ds\right]\\ =\ &\mathrm{tr}(C^\tau PC) + \limsup_{t\to\infty}\frac{1}{t}\int_0^t \|x_s - x_{s'}\|_G^2 ds = \mathrm{tr}(C^\tau PC) + O(\Delta T^2).\end{aligned}$$

Hence, (7) holds.

2 SD-based LQ optimal control of time-varying systems

In this section, we consider the SD-based optimal LQ control of LCT systems with Markovian jump parameters. The system model is as follows:

$$dx_t = A(r_t)x_t dt + B(r_t)u_t dt + C(r_t)dW_t, \tag{23}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ and $W_t \in \mathbb{R}^n$ are system state, input and disturbance, respectively; W_t is a standard Brownian motion; r_t is a continuous-time discrete-state Markov process taking values in a finite set $\mathcal{S} = \{1, \cdots, N\}$ with transition probability matrix (see ref. [10]) given by

$$P(\tau) = [P_{ij}(\tau)] = [P(r_{t+\tau} = j \mid r_t = i)] = e^{\Lambda\tau}, \quad t_0 \leqslant t \leqslant t+\tau, \tag{24}$$

where $\Lambda = (\lambda_{ij}), \lambda_{ij} \geqslant 0, j \neq i$, and

$$-\lambda_{ii} = \sum_{j=1, j\neq i}^{N} \lambda_{ij}. \tag{25}$$

Since r_t is a finite state Markov process, it can be shown that $|\lambda_{ij}| < \infty$ (see e.g. ref. [9]). Suppose that $A(r_t)$ and $B(r_t)$ are known, and that the initial values x_0 and Markov process $\{r_t\}$ are independent. Since in any finite time interval, almost all sample paths of $\{r_t\}$ are step functions with, if any, at most a finite number of discontinuous points, the solution of (23) in any finite time interval can be regarded as a finite joining of the solutions of finite time-invariant systems (see ref. [10]). For notational simplicity, in the sequel, when $r_t = i$, we will denote $A(r_t) = A_i$, $B(r_t) = B_i$, etc.

The objective of this section is to study the force of the sample step size on the control effects, to find out the stabilizability condition by the SD-based control, further to investigate the optimal index based on SD-based LQ control.

As in the above section, our objectives are (a) to investigate the effect of the sample step size on the control functions, (b) to find out the stabilizability condition by the SD-based control, and (c) to study the optimality of the SD-based LQ control.

Firstly, we introduce the following definition named Stochastic Stabilizability[11,12]. This definition was aimed at "noise free" system

$$dx_t = A(r_t)x_t dt + B(r_t)u_t dt \tag{26}$$

and (24).

Definition 2.1. We say that systems (26) and (24) are stochastic stabilizability if, for all finite $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathcal{S}$, there exists a linear feedback control law $u_t = -L(r_t)x(t)$ with $\|L(r_t)\| < \infty$ and a symmetric positive definite matrix $\widetilde{M}$ satisfying

$$\lim_{T\to\infty} E\left\{\int_0^T x^\tau(t,x_0,r_0,u)x(t,x_0,r_0,u)dt \mid x_0, r_0\right\} \leqslant x_0^\tau \widetilde{M} x_0.$$

Or we simply say that $[A(r_t), B(r_t)]$ is stochastic stabilizability.

The LQ index to be analyzed is of the following form, which is often used in stochastic systems:

$$J^0(u) = \limsup_{T\to\infty} \frac{1}{T} E\left\{\int_0^T (x^\tau(t)Q(r_t)x(t) + u^\tau(t)R(r_t)u(t))dt \mid x_0, r_0\right\}, \tag{27}$$

where $R_i > 0$, $Q_i \geqslant 0$, and $[A_i, Q_i^{1/2}]$ is observable.

As pointed out in ref. [11], for any given positive definite matrix R_i, nonnegative definite matrix Q_i, the N-coupled algebraic Riccati equation set

$$A_i^\tau M_i + M_i A_i - M_i B_i R_i^{-1} B_i^\tau M_i + \sum_{j=1}^N \lambda_{ij} M_j + Q_i = 0 \tag{28}$$

has a unique set of positive definite matrices $\{M_i,\ i = 1,\ 2,\ \cdots,\ N\}$ if and only if $[A(r_t), B(r_t)]$ is stochastic stabilizability and $[A_i, Q_i^{1/2}]$ is observable.

Similar to the last section, we introduce an admissible control set:

$$\mathcal{U}^0 = \{u:\ u_t \in \sigma\{x_s : s \leqslant t\} \text{ which makes the state}\{x_t\} \text{ of (23) satisfy}$$
$$\limsup_{t\to\infty} \frac{1}{t} E\|x_t\|^2 = 0 \text{ and } \limsup_{t\to\infty} \frac{1}{t} E\int_0^t \|x_s\|^2 ds < \infty\}. \tag{29}$$

We have the following conclusions:

Theorem 2.1. Consider the system (23)-(24). Assume $[A(r_t), B(r_t)]$ is stochastic stabilizability; and $[A_i, Q_i^{1/2}]$ $(\forall i \in \mathcal{S})$ is observable; and Markov process $\{r_t, t \geqslant 0\}$ and Brownian motion $\{W_t, t \geqslant 0\}$ are independent. Then

(i) for any $u \in \mathcal{U}^0$, $J^0(u) \geqslant \limsup_{t\to\infty} \frac{1}{t} E\int_0^t \mathrm{tr}(C^\tau(r_s)M(r_s)C(r_s))ds$;

(ii) the FS-based LQ control

$$u_t^* \triangleq -L(r_t)x_t = -R(r_t)^{-1}B(r_t)^\tau M(r_t)x_t \tag{30}$$

is in $\mathcal{U}^0$, and such that

$$J^0(u^*) = \limsup_{t\to\infty} \frac{1}{t} E \int_0^t \operatorname{tr}(C^\tau(r_s)M(r_s)C(r_s))ds = \min_{u\in\mathcal{U}^0} J^0(u),$$

where M_i, R_i, Q_i are defined in (28) and (27), respectively.

Proof. (i) Similar to (2.29) of ref. [10], by (23) we have

$$\widetilde{\mathcal{A}}(x_t^\tau M(r_t)x_t) \stackrel{\triangle}{=} \lim_{\Delta\to 0} \frac{1}{\Delta}\left(E[x_{t+\Delta}^\tau M(r_{t+\Delta})x_{t+\Delta} \mid r_t] - x_t^\tau M(r_t)x_t\right)$$
$$= x_t^\tau (A_1^\tau(r_t)M(r_t) + M(r_t)A_1(r_t) + \sum_j \lambda_{r_t,j}M_j)x_t + \operatorname{tr}(C^\tau(r_t)M(r_t)C(r_t)), \tag{31}$$

where $\widetilde{\mathcal{A}}$ is the infinitesimal operator of the joint process $\{r_t, x_t\}$. Then, from Dynkin's formula

$$\begin{aligned} Ex_t^\tau M(r_t)x_t = {} & Ex_0^\tau M(r_0)x_0 + E\int_0^t x_s^\tau(A^\tau(r_s)M(r_s) + M(r_s)A(r_s) + \sum_j \lambda_{r_s,j}M_j)x_s ds \\ & + E\int_0^t (u_s^\tau B^\tau(r_s)M(r_s)x_s + x_s^\tau M(r_s)B(r_s)u_s)ds \\ & + E\int_0^t \operatorname{tr}(C^\tau(r_s)M(r_s)C(r_s))ds. \end{aligned}$$

And hence, we have

$$\begin{aligned} & E\left\{x_t^\tau M(r_t)x_t + \int_0^t (x^\tau(s)Q(r_s)x(s) + u^\tau(s)R(r_s)u(s))ds\right\} \\ & = Ex_0^\tau M(r_0)x_0 + E\int_0^t \operatorname{tr}(C(r_s)^\tau M(r_s)C(r_s))ds \\ & \quad + E\int_0^t (u_s + R(r_s)^{-1}B(r_s)^\tau M(r_s)x_s)^\tau R(r_s)(u_s + R(r_s)^{-1}B(r_s)^\tau M(r_s)x_s)ds. \end{aligned} \tag{32}$$

So, for any $u \in \mathcal{U}^0$, we get

$$\begin{aligned} & J^0(u) \\ & = \limsup_{t\to\infty} \frac{1}{t} E\int_0^t (x_s^\tau Q(r_s)x_s + u_s^\tau R(r_s)u_s)ds \\ & = \limsup_{t\to\infty} \left\{\frac{1}{t}E\int_0^t (u_s + R(r_s)^{-1}B(r_s)^\tau M(r_s)x_s)^\tau R(r_s)(u_s + R(r_s)^{-1}B(r_s)^\tau M(r_s)x_s)ds \right. \\ & \quad \left. + \frac{1}{t}E\int_0^t \operatorname{tr}(C(r_s)^\tau M(r_s)C(r_s))ds\right\} \geqslant \limsup_{t\to\infty} \frac{1}{t}E\int_0^t \operatorname{tr}(C(r_s)^\tau M(r_s)C(r_s))ds. \end{aligned}$$

Here we have used inequations: $\lim_{t\to\infty} \frac{1}{t}E\|x_t\|^2 = 0$ and $\limsup_{t\to\infty} \frac{1}{t}E\int_0^t \|x_s\|^2 ds < \infty$.

(ii) Under the control (30), system (23) becomes

$$dx_t = A_1(r_t)x_t dt + C(r_t)dW_t, \tag{33}$$

where

$$A_1(r_t) \stackrel{\triangle}{=} A(r_t) - B(r_t)L(r_t). \tag{34}$$

Recalling $[A(r_t), B(r_t)]$ is stochastic stabilizability, from ref. [11] we know that the symmetric solutions, K_i $(i \in \mathcal{S})$, of the equation set

$$A_{1,i}^\tau K_i + K_i A_{1,i} + \sum_j \lambda_{i,j}K_j = -I \tag{35}$$

are positive definite. Construct $K(r_t)$ such that $K(r_t) = K_i$ when $r_t = i$. Thus, similar to (2.29) of ref. [10], we have

$$\begin{aligned}\widetilde{\mathcal{A}}(x_t^\tau K(r_t)x_t) &\triangleq \lim_{\Delta\to 0}\frac{1}{\Delta}\left(E[x_{t+\Delta}^\tau K(r_{t+\Delta})x_{t+\Delta} \mid r_t] - x_t^\tau K(r_t)x_t\right)\\ &= x_t^\tau(A_1^\tau(r_t)K(r_t) + K(r_t)A_1(r_t) + \sum_j \lambda_{r_t,j}K_j)x_t + \mathrm{tr}(C^\tau(r_t)K(r_t)C(r_t))\\ &= -\|x_t\|^2 + \mathrm{tr}(C^\tau(r_t)K(r_t)C(r_t)),\end{aligned} \tag{36}$$

where $\widetilde{\mathcal{A}}$ is the infinitesimal operator of the joint process $\{r_t, x_t\}$. Then, from Dynkin's formula

$$E\left\{x_t^\tau K(r_t)x_t + \int_0^t \|x_s\|^2 ds\right\} = Ex_0^\tau K(r_0)x_0 + E\int_0^t \mathrm{tr}(C^\tau(r_s)K(r_s)C(r_s))ds. \tag{37}$$

Therefore,

$$E\|x_t\|^2 = O(t) \quad \text{and} \quad E\int_0^t \|x_s\|^2 ds = O(t). \tag{38}$$

We now show

$$\limsup_{t\to\infty} E\|x_t\|^2 < \infty. \tag{39}$$

Since for each $i \in \mathcal{S}$, $K(i) > 0$, and $\mathcal{S}$ has only finite elements, there exist constants $\alpha > \beta > 0$ such that

$$0 < \beta I \leqslant K(r_t) \leqslant \alpha I, \quad \forall t \geqslant 0. \tag{40}$$

This together with (36) implies

$$\widetilde{\mathcal{A}}(x_t^\tau K(r_t)x_t) \leqslant -\frac{1}{\alpha}x_t^\tau K(r_t)x_t + \mathrm{tr}(C^\tau(r_t)K(r_t)C(r_t)),$$

or

$$Ex_t^\tau K(r_t)x_t \leqslant e^{-\frac{1}{\alpha}t}Ex_0^\tau K(r_0)x_0 + 2E\int_0^t e^{-\frac{1}{\alpha}(t-s)}\mathrm{tr}(C^\tau(r_s)K(r_s)C(r_s))ds = O(1). \tag{41}$$

From this and (40) we see that (39) holds. Hence $u^* \in \mathcal{U}^0$.

By (39) and (32), we obtain

$$J^0(u^*) = \limsup_{t\to\infty}\frac{1}{t}E\int_0^t \mathrm{tr}(C^\tau(r_s)M(r_s)C(r_s))ds.$$

From now on, we will focus on the effect analysis of the SD-based LQ control law:

$$u_t = -L(i)x_t = -R_i^{-1}B_i^\tau M_i x_{k\Delta T}, \quad t \in [k\Delta T, (k+1)\Delta T), \quad r_{k\Delta T} = i. \tag{42}$$

To this end, set

$$h_1 = \max_{i\in\mathcal{S}}\|\bar{A}_1(i)\|, \quad h = \max_{i\in\mathcal{S}}\|A_i\|, \quad d_1 = \Delta T e^{h\Delta T}h_1, \tag{43}$$

$$\bar{A}_1(r_t) = A(r_t) - B(r_t)L(r_{t'}), \quad t' = \left\lceil \frac{t}{\Delta T}\right\rceil \Delta T, \tag{44}$$

$$L = \max_{i\in\mathcal{S}}\|M_iB_iR_i^{-1}\| \cdot \max_{j\in\mathcal{S}}\|B_j^\tau K_j\|, \quad \lambda = (N-1)(e^{\|A\|}-1), \tag{45}$$

where K_j is the solution of eq. (35).

Theorem 2.2. Consider system (23)-(24). Assume $[A(r_t), B(r_t)]$ is stochastic stabilizability; $[A_i, Q_i^{1/2}]$ ($\forall i \in \mathcal{S}$) is observable; and Markov process $\{r_t, t \geqslant 0\}$ and Brownian motion

$\{W_t, t \geqslant 0\}$ are independent. Then under (42), if the sample step size ΔT satisfies

$$\begin{cases} d_1 \triangleq \Delta T e^{h\Delta T} h_1 < 1, \quad \Delta T \leqslant 1, \\ \dfrac{Ld_1}{1-d_1} + \dfrac{8(1+d_1)^2}{(1-d_1)^2} L\lambda\Delta T \leqslant \dfrac{1}{3}, \end{cases} \tag{46}$$

then

$$\limsup_{t\to\infty} \frac{1}{t} E \int_0^t (\|x_s\|^2 + \|u_s\|^2) ds < \infty \tag{47}$$

and

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} E \int_0^t (x_s^\tau Q(r_s) x_s + u_s^\tau R(r_s) u_s) ds \\ \leqslant\ &\limsup_{t\to\infty} \frac{1}{t} E \int_0^t \mathrm{tr}(C^\tau(r_s) M(r_s) C(r_s)) ds + O(\Delta T). \end{aligned} \tag{48}$$

Remark 2.1. Similar to Remarks 1.2 and 1.3, a practical range of the sample step size ΔT can be easily determined if our objective is only to stabilize the system and get a suboptimal or satisfactory SD-based LQ control, whereas it may be difficult and complex to clarify the maximum range of such ΔT or choose ΔT optimally in some sense.

We now introduce a lemma before proving Theorem 2.2.

Lemma 2.1. Suppose r_t is a Markov process taking values in a finite set $\mathcal{S} = \{1, \cdots, N\}$, and is subject to (24) and (25); and $f(r_s)$: $\mathbb{R}^1 \to \mathbb{R}^1$ and $g(r_s)$: $\mathbb{R}^1 \to \mathbb{R}^{n\times n}$ are measurable functions with respect to $\sigma\{r_t, t \geqslant s\}$. Then, in the case where $s - s_0 \leqslant \Delta T$ and $\Delta T \leqslant 1$,

$$E\left[\|g(r_s) - g(r_{s_0})\| \int_{s_0}^s f(r_\mu) d\mu \mid r_{s_0} = i\right] \leqslant \max_{j\neq i} \|g(j) - g(i)\| \max_{l\in\mathcal{S}} f(l) \cdot \lambda(s-s_0)^2,$$

where $\lambda = (N-1)(e^{\|\Lambda\|} - 1)$.

Proof of Lemma 2.1. When $s - s_0 \leqslant \Delta T$ and $\Delta T \leqslant 1$, it is easy to see

$$\sum_{j\neq i} P_{ij}(s-s_0) \leqslant (s-s_0) \sum_{j\neq i} \left(\frac{e^{\Lambda(s-s_0)} - I}{s-s_0}\right)_{ij} \leqslant (s-s_0) \sum_{j\neq i} \left\|\frac{e^{\Lambda(s-s_0)} - I}{s-s_0}\right\| \leqslant \lambda(s-s_0). \tag{49}$$

Since in any finite time interval, almost all sample paths of the Markov process $\{r_t\}$ are step functions with at most a finite number of discontinuous points, suppose these discontinuous points are $s_1, s_2, \cdots, s_m$, and satisfy $s_0 < s_1 < s_2 < \cdots < s_m < s_{m+1} = s$. Then by (49) we obtain

$$\begin{aligned} &E\left[\|g(r_s) - g(r_{s_0})\| \int_{s_0}^s f(r_\mu) d\mu \mid r_{s_0} = i\right] \\ = &\sum_{\substack{r_s = j, \\ j \neq i}} \|g(j) - g(i)\| \sum_{r_{s_1}\in\mathcal{S},\cdots,r_{s_m}\in\mathcal{S}} \sum_{i=0}^m f(r_{s_i})(s_{i+1} - s_i) \prod_{i=0}^m P_{r_{s_i}, r_{s_{i+1}}}(s_{i+1} - s_i) \\ \leqslant &\sum_{\substack{r_s = j, \\ j \neq i}} \|g(j) - g(i)\| \left(\max_{l\in\mathcal{S}} \sum_{i=0}^m f(l)(s_{i+1} - s_i)\right) \left(\sum_{r_{s_1}\in\mathcal{S},\cdots,r_{s_m}\in\mathcal{S}} \prod_{i=0}^m P_{r_{s_i}, r_{s_{i+1}}}(s_{i+1} - s_i)\right) \end{aligned}$$

$$\leqslant \sum_{\substack{r_s = j,\\ j \neq i}} \|g(j) - g(i)\| \max_{l\in\mathcal{S}} f(l)(s - s_0) \cdot P_{ij}(s - s_0) \leqslant \lambda(s - s_0)^2 \cdot \max_{j\neq i} \|g(j) - g(i)\| \max_{l\in\mathcal{S}} f(l).$$

Proof of Theorem 2.2. Define

$$A_1(r_t) \triangleq A(r_t) - B(r_t)L(r_t), \quad B_1(r_t) \triangleq B(r_t)L(r_t), \quad \bar{B}_1(r_t) \triangleq B(r_t)L(r_{t'}),$$
$$u_t^* = -L(r_t)x_t = -R(r_t)^{-1}B(r_t)^\tau M(r_t)x_t.$$

Then system (23) with SD-based control law (42) has the following form:

$$\begin{aligned} dx_t &= A_1(r_t)x_t dt + B(r_t)(u_t - u_t^*)dt + C(r_t)dW_t \\ &= \bar{A}_1(r_t)x_{t'}dt + A(r_t)(x_t - x_{t'})dt + C(r_t)dW_t. \end{aligned} \tag{50}$$

This gives

$$x_t - x_{t'} = \int_{t'}^t A(r_s)(x_s - x_{t'})ds + \int_{t'}^t \bar{A}_1(r_s)ds \cdot x_{t'} + \int_{t'}^t C(r_s)dW_s. \tag{51}$$

Hence, we have

$$\|x_t - x_{t'}\| \leqslant \int_{t'}^t \|A(r_s)\| \cdot \|x_s - x_{t'}\|ds + \int_{t'}^t \|\bar{A}_1(r_s)\|ds \cdot \|x_{t'}\| + \|\int_{t'}^t C(r_s)dW_s\|. \tag{52}$$

Applying Grownwall lemma, we obtain

$$\begin{aligned} \|x_t - x_{t'}\| \leqslant{}& \int_{t'}^t \|\bar{A}_1(r_s)\|ds \cdot \|x_{t'}\| \cdot e^{\int_{t'}^t \|A(r_t)\|ds} + \left\|\int_{t'}^t C(r_s)dW_s\right\| \\ &+ \int_{t'}^t h \cdot e^{h(t-s)} \cdot \left\|\int_{t'}^s C(r_\mu)dW_\mu\right\| ds \leqslant d_1\|x_{t'}\| + d_2(t), \end{aligned} \tag{53}$$

where d_1 is given by (43), and

$$d_2(t) = he^{h\Delta T}\int_{t'}^t \left\|\int_{t'}^s C(r_\mu)dW_\mu\right\| ds + \left\|\int_{t'}^t C(r_s)dW_s\right\|. \tag{54}$$

Similar to refs. [10, 11], by (50), Ito's formula and (35), we get

$$\begin{aligned} Ex_t^\tau K(r_t)x_t ={}& Ex_0^\tau K(r_0)x_0 + E\int_0^t x_s^\tau(A_1^\tau(r_s)K(r_s) + K(r_s)A_1(r_s) + \sum_j \lambda_{r_s,j}K_j)x_s ds \\ &+ 2E\int_0^t (u_s - u_s^*)^\tau B^\tau(r_s)K(r_s)x_s + E\int_0^t \mathrm{tr}(C^\tau(r_s)K(r_s)C(r_s))ds \\ ={}& Ex_0^\tau K(r_0)x_0 - E\int_0^t \|x_s\|^2 ds + E\int_0^t \mathrm{tr}(C^\tau(r_s)K(r_s)C(r_s))ds \\ &+ 2E\int_0^t (u_s - u_s^*)^\tau B^\tau(r_s)K(r_s)x_s ds. \end{aligned} \tag{55}$$

Notice that

$$\begin{aligned} &(u_s - u_s^*)^\tau B^\tau(r_s)K(r_s)x_s = (L(r_s)x_s - L(r_{s'})x_{s'})^\tau B^\tau(r_s)K(r_s)x_s \\ &= (x_s - x_{s'})^\tau L^\tau(r_{s'})B^\tau(r_s)K(r_s)x_s + x_s^\tau(L(r_s) - L(r_{s'}))^\tau B^\tau(r_s)K(r_s)x_s. \end{aligned} \tag{56}$$

The first term has the following upperbound estimate

$$\begin{aligned} &(x_s - x_{s'})^\tau L^\tau(r_{s'})B^\tau(r_s)K(r_s)x_s \\ &\leqslant L \cdot \|x_s - x_{s'}\| \cdot \|x_s\| \leqslant L \cdot [d_1\|x_{s'}\| + d_2(s)] \cdot \|x_s\| \\ &\leqslant L \cdot \left[\frac{d_1}{1 - d_1}\|x_s\| + \frac{d_2(s)}{1 - d_1}\right] \cdot \|x_s\| \leqslant \alpha_2\|x_s\|^2 + \beta_2(s)\|x_s\|, \end{aligned} \tag{57}$$

where

$$\alpha_2 = \frac{Ld_1}{1-d_1}, \quad \beta_2(t) = \frac{Ld_2(t)}{1-d_1}. \tag{58}$$

Here we have used the condition $d_1 < 1$, (53) and the following inequality induced from (53):

$$\|x_{s'}\| \leqslant \frac{1}{1-d_1}\|x_s\| + \frac{d_2(s)}{1-d_1}. \tag{59}$$

Let $c = \max\limits_{i\in\mathcal{S}} \mathrm{tr}(C_i^\tau C_i)$. Then, by (54), we have

$$\begin{aligned}
Ed_2(t)^2 &\leqslant 2E\left\|\int_{t'}^t C(r_s)dW_s\right\|^2 + 2h^2e^{2h\Delta T}(t-t')\int_{t'}^t E\left\|\int_{t'}^s C(r_\mu)dW_\mu\right\|^2 ds \\
&= 2E\int_{t'}^t \mathrm{tr}(C^\tau(r_s)C(r_s))ds + 2h^2e^{2h\Delta T}(t-t')\int_{t'}^t E\int_{t'}^s \mathrm{tr}(C^\tau(r_\mu)C(r_\mu))d\mu ds \\
&\leqslant 2c(t-t') + ch^2e^{2h\Delta T}(t-t')^3.
\end{aligned}$$

Therefore, from $\Delta T \leqslant 1$ it follows

$$E\int_{k\Delta T}^{(k+1)\Delta T} d_2^2(s)ds \leqslant c\Delta T^2 + \frac{1}{4}ch^2e^{2h\Delta T}\Delta T^4 \leqslant c\Delta T^2(1+h^2e^{2h}). \tag{60}$$

From this and Lemma 2.1, integrating the second term of (56) on interval $(0, K\Delta T]$ $(K \in I\!N)$, and then, taking the expectation, we have

$$\begin{aligned}
&E\int_0^{K\Delta T} x_s^\tau(L(r_s) - L(r_{s'}))^\tau B^\tau(r_s)K(r_s)x_s ds \\
&= E\int_0^{K\Delta T} \|(L(r_s) - L(r_{s'}))^\tau B^\tau(r_s)K(r_s)\| \cdot \left(2(1+d_1)^2\|x_{s'}\|^2 + 2d_2(s)^2\right) ds \\
&\leqslant \sum_{k=0}^{K-1}\sum_{i=1}^{N} E\left[\int_{k\Delta T}^{(k+1)\Delta T} \|(L(r_s) - L(r_{s'}))^\tau B^\tau(r_s)K(r_s)\| \cdot 2(1+d_1)^2\|x_{k\Delta T}\|^2 ds \mid r_{k\Delta T} = i\right] \\
&\quad \cdot P(r_{k\Delta T} = i) + \sum_{k=0}^{K-1} E\int_{k\Delta T}^{(k+1)\Delta T} \|(L(r_s) - L(r_{k\Delta T}))^\tau B^\tau(r_s)K(r_s)\| \cdot 2d_2(s)^2 ds \\
&\leqslant \frac{8(1+d_1)^2}{(1-d_1)^2}L\lambda\Delta T\sum_{k=0}^{K-1} E\int_{k\Delta T}^{(k+1)\Delta T}(\|x_s\|^2 + d_2(s)^2)ds + 4L\sum_{k=0}^{K-1} E\int_{k\Delta T}^{(k+1)\Delta T} d_2^2(s)ds \\
&\leqslant \frac{8(1+d_1)^2}{(1-d_1)^2}L\lambda\Delta T E\int_0^{K\Delta T}\|x_s\|^2 ds + 4cL\Delta T(1+h^2e^{2h})\left(1 + \frac{2(1+d_1)^2}{(1-d_1)^2}\lambda\right)K\Delta T.
\end{aligned} \tag{61}$$

Here we have used $\Delta T \leqslant 1$ and the following inequality induced from (53):

$$\|x_s\|^2 \leqslant 2(1+d_1)^2\|x_{s'}\|^2 + 2d_2(s)^2, \quad \|x_{s'}\|^2 \leqslant \frac{2}{(1-d_1)^2}\|x_s\|^2 + \frac{2d_2(s)^2}{(1-d_1)^2}. \tag{62}$$

Similarly, it can be shown that (61) holds for any $t \geqslant 0$, i.e.

$$\begin{aligned}
&E\int_0^t x_s^\tau(L(r_s) - L(r_{s'}))^\tau B^\tau(r_s)K(r_s)x_s ds \\
&\leqslant \frac{8(1+d_1)^2}{(1-d_1)^2}L\lambda\Delta T E\int_0^t \|x_s\|^2 ds + 4Lc\Delta T(1+h^2e^{2h})\left(1 + \frac{2(1+d_1)^2}{(1-d_1)^2}\lambda\right)t.
\end{aligned} \tag{63}$$

Furthermore, by (55), (56), (57), (63) and the Schwartz inequality, we have

$$\begin{aligned}
&Ex_t^\tau K(r_t)x_t \\
\leqslant\ & Ex_0^\tau K(r_0)x_0 - \left(1 - 2\alpha_2 - \frac{16(1+d_1)^2}{(1-d_1)^2}L\lambda\Delta T\right) E\int_0^t \|x_s\|^2 ds \\
&+ E\int_0^t \operatorname{tr}(C^\tau(r_s)K(r_s)C(r_s))ds + 2E\int_0^t \beta_2(s)\|x_s\|ds \\
&+ 8cL\Delta T\left(1+h^2e^{2h}\right)\left(1+\frac{2(1+d_1)^2}{(1-d_1)^2}\lambda\right)t \\
\leqslant\ & Ex_0^\tau K(r_0)x_0 - \left(1 - 2\alpha_2 - \frac{16(1+d_1)^2}{(1-d_1)^2}L\lambda\Delta T\right) E\int_0^t \|x_s\|^2 ds \\
&+ E\int_0^t \operatorname{tr}(C^\tau(r_s)K(r_s)C(r_s))ds + 2E\left(\int_0^t \beta_2(s)^2 ds\right)^{1/2}\left(\int_0^t \|x_s\|^2 ds\right)^{1/2} \\
&+ 8cL\Delta T\left(1+h^2e^{2h}\right)\left(1+\frac{2(1+d_1)^2}{(1-d_1)^2}\lambda\right)t.
\end{aligned}$$

From (46) it is easy to verify

$$1 - 2\alpha_2 - \frac{16(1+d_1)^2}{(1-d_1)^2}L\lambda\Delta T \geqslant \frac{1}{3}.$$

Noticing the independent increment property of the standard Brownian motion $\|W_t - W_{t'}\|$, similar to (60), by the large number law we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \beta_2(s)^2 ds = O(\Delta T^2).$$

Hence,

$$\begin{aligned}
&\limsup_{t\to\infty} E\left(\frac{1}{t}x_t^\tau K(r_t)x_t + \left(1 - 2\alpha_2 - \frac{16(1+d_1)^2}{(1-d_1)^2}L\lambda\Delta T\right)\frac{1}{t}\int_0^t \|x_s\|^2 ds\right) \\
\leqslant\ & \limsup_{t\to\infty}\frac{1}{t}E\int_0^t \operatorname{tr}(C^\tau(r_s)K(r_s)C(r_s))ds + O(\Delta T) \\
&+ O\left(\Delta T\cdot\limsup_{t\to\infty}\left(\frac{1}{t}E\int_0^t \|x_s\|^2 ds\right)^{1/2}\right).
\end{aligned}\tag{64}$$

Thus,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t \|x_s\|^2 ds < \infty, \tag{65}$$

i.e. (47) holds.

Now we calculate the quadratic index. By (59) and (65) we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t E\|x_{s'}\|^2 ds < \infty. \tag{66}$$

Then, similar to (61), by (53) we get

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}E\int_0^t (u_s - u_s^*)^\tau R(r_s)(u_s - u_s^*)ds \\
=\ & \limsup_{t\to\infty}\frac{1}{t}E\int_0^t [x_s^\tau(L(r_s) - L(r_{s'}))^\tau R(r_s)(L(r_s) - L(r_{s'}))x_s \\
&+ 2x_s^\tau(L(r_s) - L(r_{s'}))^\tau R(r_s)L(r_{s'})(x_s - x_{s'}) \\
&+(x_s - x_{s'})^\tau L^\tau(r_{s'})R(r_s)L(r_{s'})(x_s - x_{s'})]\, ds = O(\Delta T).
\end{aligned}\tag{67}$$

Therefore, similar to (14), by the coupled Riccati equation (28) we have

$$\begin{aligned}
&\limsup_{t\to\infty} E\left(\frac{1}{t}x_t^\tau M(r_t)x_t + \frac{1}{t}\int_0^t (x_s^\tau Q(r_s)x_s + u_s^\tau R(r_s)u_s)ds\right)\\
&= \limsup_{t\to\infty} \frac{1}{t}E\int_0^t \mathrm{tr}(C^\tau(r_s)M(r_s)C(r_s))ds + \limsup_{t\to\infty}\frac{1}{t}E\int_0^t (u_s-u_s^*)^\tau R(r_s)(u_s-u_s^*))ds\\
&= \limsup_{t\to\infty} \frac{1}{t}E\int_0^t \mathrm{tr}(C^\tau(r_s)M(r_s)C(r_s))ds + O(\Delta T).
\end{aligned} \tag{68}$$

3 Conclusions

General stochastic LCT systems are studied in this work, including those with time-invariant or time-varying parameters. The control design is directly based on the original continuous system and the original continuous performance index, without involving any discretized models or discretized indices. In addition to stability analysis of the closed-loop systems, optimality of performance index of the closed-loop system is analyzed. Particularly, the performance indices corresponding to the SD-based LQ control and the FS-based LQ control, respectively, are compared. And the difference between the performance indices is quantitatively expressed. It is shown that when sample time ΔT is small, so is the index difference. Besides, the upper bounds of the differences are also presented, which are $O(\Delta T^2)$ and $O(\Delta T)$, respectively. As for how to figure out the maximal range of ΔT, how to choose ΔT optimally, and how to obtain an explicit expression describing the relationship between the sample step size and system structure and parameters, it is very difficult and complex, and should be analyzed case by case.

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